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The Dual of a Logical Linear Programme

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Abstract. A Linear Programme (LP) involves a *conjunction* of linear constraints and has a well defined dual. It is shown that if we allow the full set of Boolean connectives $\{\wedge, \vee, \sim\}$ applied to a set of linear constraints we get a model which we define as a Logical Linear Programme (LLP). This also has a well defined dual preserving most of the properties of LP duality. Generalisations of the connectives are also considered together with the relationship with Integer Programming formulation.

Key words: Logic; Linear programming; Boolean algebra; Duality

1. Introduction

A Linear Programme (LP) involves maximising or minimising a linear function of real numbers subject to a *conjunction* of linear constraints. It is well known that any LP model (the Primal model) has a corresponding Dual model. The optimal solutions to the primal and dual models are intimately related giving rise to the duality and complementarity theorems (see e.g. Dantzig [5]).

In Section 2 we generalise the concept of an LP model by allowing any Boolean function of the constraints. We can restrict ourselves to the complete set of connectives *and* ($\wedge$), *or* ($\vee$), *not* ($\sim$). Individual linear constraints can be regarded as atomic propositions which get combined into compound propositions by these connectives. We call the resultant model a *Logical Linear Programme* (LLP). Since the negation of standard LP constraints results in strict inequality relations we have to allow optimisation over open regions and must replace maximisation and minimisation by the operations of supremum and infimum respectively. For practical purposes (and when negation is not involved) we can approximate '$<$' and '$>$' constraints by '$\leqslant$' and '$\geqslant$' and the modification is not necessary. It is shown in Section 4, every LLP has a natural dual and, in Section 5, that (with one restriction) a corresponding duality theorem to that of LP holds.

A natural notation to use, which preserves the analogy with LP, is that of Minimax Algebra (see Carré [3] and Cuninghame-Green [4]).

In Section 3 we give some illustrative numerical examples and give their duals, together with their solutions in Section 6.

In Section 7 Boolean Connectives are generalised and the dual model further extended.

The conventional way of dealing with logical conditions applied to linear constraints is by the use of 0-1 integer variables. This results in a mixed integer linear programme (MIP) which is a conjunction of constraints involving real and 0-1 variables. Such a model can obscure the logical relationships implicit in the original problem which could be exploited in the solution process. Also a satisfactory definition of a dual for a MIP does not yet exist. The relationship between LLPs and MIPs is discussed in Section 8.

2. A logical linear programme

We define an LLP as a model of the form

$$z_L = \text{Supremum} \sum_{j\in J} c_j x_j$$
$$P: \quad \text{subject to:} \quad L\Big(\sum_{j\in J} a_{ij}x_j \cdot \rho_i \cdot b_i\Big), \quad x_j \in \mathbb{R}^+$$

where $i \in I$, $\rho_i \in \{\leqslant, \geqslant, =\}$.

L is a compound proposition made up of the atomic propositions $\sum_{j\in J} a_{ij}x_j \cdot \rho_i \cdot b_i$ from the connectives '$\wedge$', '$\vee$', and '$\sim$'.

If we wish to find the Infimum it is convenient to keep the model in the form above and negate the objective function. Also it is convenient to restrict the x_j to be explicitly non-negative rather than do this by means of constraints. Variants to this representation and the allowing of free (unrestricted) variables is straightforward.

LP is the special case in which L is $\wedge_i$. 'Supremum' (Sup) is then equivalent to 'Maximum' and 'Infimum' (Inf) is equivalent to 'Minimum'.

We will allow L to be a general Boolean function involving the connectives '$\wedge$','$\vee$' and '$\sim$' and give the dual of P. However, it is sometimes convenient to express L in Disjunctive Normal Form (DNF) or Conjunctive Normal Form (CNF). In DNF a model takes the form

$$\text{Supremum} \sum_{j\in J} c_j x_j$$
$$\text{PA:} \quad \text{subject to:} \quad \bigvee_k \bigwedge_i \sum_{j\in J} a_{kij}x_j \cdot \rho_{ki} \cdot b_{ki}, \quad x_j \in \mathbb{R}^+$$

where $i \in I_k$, $k \in K$, $\rho_{ki} \in \{\leqslant, \geqslant, =, <, >\}$.

The strict inequalities '$<$' and '$>$' arise since

$$\sim\Big(\sum_j a_j x_j \leqslant b\Big) \text{ is represented as } \sum_j a_j x_j > b$$
$$\sim\Big(\sum_j a_j x_j \geqslant b\Big) \text{ is represented as } \sum_j a_j x_j < b$$
$$\sim\Big(\sum_j a_j x_j = b\Big) \text{ is represented as } \sum_j a_j x_j < b \vee \sum_j a_j x_j > b$$

In CNF a model takes the form

$$\text{Supremum} \sum_{j \in J} c_j x_j$$

$$\text{PB:} \quad \text{subject to:} \quad \bigwedge_k \bigvee_i \sum_j a_{kij} x_j \cdot \rho_{ki} \cdot b_{ki}, \quad x_j \in \mathbb{R}^+$$

where $i \in I_k$, $k \in K$, $\rho_{ki} \in \{\leqslant, \geqslant, =, <, >\}$.

If a strict inequality is binding in an optimal solution (in the sense that if removed the optimal solution would change) then it only makes sense to speak of the supremum (infimum) of at least one of the variables with a positive (negative) coefficient in a '$<$' ('$>$') constraint, rather than the variable itself. While, in this case, the model may have an optimal 'solution' in the sense of a finite supremum or infimum of the objective there is no sense of all the variables having real-number solution values giving rise to this objective value. A model will be said to be 'feasible' if there exist values of the variable, or if appropriate, their suprema or infima satisfying the constraints.

If the '~' connective does not arise then PA and PB are both forms of *Disjunctive Programmes* and again 'Supremum' and 'Infimum' can be replaced by 'Maximum' and 'Minimum'. In order to illustrate these forms of an LLP we consider a number of examples.

3. Illustrative examples

EXAMPLE 1.

$$\text{Supremum} \quad x_1 + 2x_2$$

$$\text{Subject to:} \quad \begin{bmatrix} & 2x_1 + 3x_2 \leqslant 8 \\ \wedge & \\ & \sim(x_1 - 2x_2 \leqslant 2) \end{bmatrix}$$

$$\text{E1.1} \quad \vee$$

$$\begin{bmatrix} & 3x_1 + 4x_2 \leqslant 8 \\ \wedge & \\ & \sim(4x_1 - x_2 \leqslant 6) \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

This model is already in DNF. We can alternatively express all 'atomic' inequalities strictly, giving the constraints as

$$\begin{bmatrix} & 2x_1 + 3x_2 \leqslant 8 \\ \wedge & \\ & -x_1 + 2x_2 < -2 \end{bmatrix}$$

$$\text{E1.2} \quad \vee$$

$$\begin{bmatrix} & 3x_1 + 4x_2 \leqslant 8 \\ \wedge & \\ & -4x_1 + x_2 < -6 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

or, in CNF as

$$\begin{bmatrix} & 2x_1 + 3x_2 \leqslant 8 \\ \vee & \\ & 3x_1 + 4x_2 \leqslant 8 \end{bmatrix}$$

$$\wedge$$

$$\begin{bmatrix} & 2x_1 + 3x_2 \leqslant 8 \\ \vee & \\ & -4x_1 + x_2 < -6 \end{bmatrix}$$

E1.3 $\wedge$

$$\begin{bmatrix} & -x_1 + 2x_2 < -2 \\ \vee & \\ & 3x_1 + 4x_2 \leqslant 8 \end{bmatrix}$$

$$\wedge$$

$$\begin{bmatrix} & -x_1 + 2x_2 < -2 \\ \vee & \\ & -4x_1 + x_2 < -6 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

EXAMPLE 2.

Maximise $x_1 + x_2$

Subject to:

$$\begin{bmatrix} & 2x_1 + 3x_2 \leqslant 10 \\ \vee & \\ & 3x_1 - x_2 \leqslant 11 \end{bmatrix}$$

E2.1 $\wedge$

$$\begin{bmatrix} & x_1 + 2x_2 \leqslant 6 \\ \vee & \\ & 4x_1 + x_2 \leqslant 20 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

This model is in CNF. Expressed in DNF the constraints become

$$\begin{bmatrix} & 2x_1 + 3x_2 \leqslant 10 \\ \wedge & \\ & x_1 + 2x_2 \leqslant 6 \end{bmatrix}$$

$$\vee$$

$$\begin{bmatrix} & 2x_1 + 2x_2 \leqslant 10 \\ \wedge & \\ & 4x_1 + x_2 \leqslant 20 \end{bmatrix}$$

E2.2 $\vee$

$$\begin{bmatrix} & 3x_1 - x_2 \leqslant 11 \\ \wedge & \\ & x_1 + 2x_2 \leqslant 6 \end{bmatrix}$$

$\vee$

$$\begin{bmatrix} & 3x_1 - x_2 \leqslant 11 \\ \wedge & \\ & 4x_1 + x_2 \leqslant 20 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

EXAMPLE 3.

Maximise $\quad x_1 + x_2$

Subject to:

$$\begin{bmatrix} & -2x_1 + x_2 \leqslant -1 \\ \wedge & \\ & x_1 \qquad \leqslant -1 \end{bmatrix}$$

E3.1 $\vee$

$$\begin{bmatrix} & 2x_1 + x_2 \leqslant 2 \\ \vee & \\ & x_1 - x_2 \leqslant 1 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

This model is in DNF. In CNF the constraints are

$$\begin{bmatrix} & -2x_1 + x_2 \leqslant -1 \\ \vee & \\ & 2x_1 + x_2 \leqslant 2 \end{bmatrix}$$

$\wedge$

$$\begin{bmatrix} & -2x_1 + x_2 \leqslant -1 \\ \wedge & \\ & x_1 - x_2 \leqslant 2 \end{bmatrix}$$

E3.2 $\wedge$

$$\begin{bmatrix} & x_1 \qquad \leqslant -1 \\ \vee & \\ & 2x_1 + x_2 \leqslant 2 \end{bmatrix}$$

$\wedge$

$$\begin{bmatrix} & x_1 \qquad \leqslant -1 \\ \vee & \\ & x_1 - x_2 \leqslant 1 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

4. The dual of a logical linear program

It is convenient to introduce the following notation based on Carré [3] and Cuninghame-Green [4]

$$\text{Min}\,(a, b) \text{ is written } a \oplus b$$
$$\text{Max}\,(a, b) \text{ is written } a \oplus' b$$
$$\underset{i \in I}{\text{Min}}\,(a_i) \text{ is written } \sum_{i \in I} {}_{\oplus}\, a_i$$
$$\underset{i \in I}{\text{Max}}\,(a_i) \text{ is written } \sum_{i \in I} {}_{\oplus'}\, a_i$$

Associated with each of the atomic constraints in *P*, *PA* and *PB* we have a suitably restricted *dual variable* y_i.

If ρ_i is $\leqslant$ $\quad y_i \geqslant 0$
If ρ_i is $\geqslant$ $\quad y_i \leqslant 0$
If ρ_i is $<$ $\quad y_i \geqslant 0$ and if $\text{Sup}\,(y_i) > 0$ then $y_i \neq \text{Sup}\,(y_i)$
If ρ_i is $>$ $\quad y_i \leqslant 0$ and if $\text{Inf}\,(y_i) < 0$ then $y_i \neq \text{Inf}\,(y_i)$

In the case of '$<$' ('$>$') inequalities it only makes sense to speak of either the dual values being zero or the supremum (infimum) of these dual values being non-zero. In practice strict inequalities would be transformed to non-strict ones and the problem would not arise.

The dual of *P* is defined recursively by mapping the Boolean function *L* into an arithmetic function *F* by assuming L_1 and L_2 have already been mapped into F_1 and F_2 respectively.

$$L_1 \wedge L_2 \rightarrow F_1 + F_2$$
$$L_1 \vee L_2 \rightarrow F_1 \oplus F_2$$
$$\sim L_1 \rightarrow -\text{Supremum } F_1$$

L_ℓ is a Boolean function of the statements $\Sigma_{j \in J}\, a_{ij}x_j \cdot \rho_i \cdot b_i$ for $i \in I_\ell$.
F_ℓ is an arithmetic function defined on a_{ij} and b_i for $i \in I_\ell$.
If *L* maps to *F* we also define F' obtained from *F* by replacing all occurrences of $\oplus$ by $\oplus'$.

The dual of *P* is defined as

$$z_F = \text{Infimum}\, F'(b_1 y_1, b_2 y_2, \ldots, b_m y_m)$$
$$D\text{:} \quad \text{subject to:} \quad F(a_{1j} y_1, a_{2j} y_2, \ldots, a_{mj} y_m) \geqslant c_j \text{ all } j \in J$$

where we define *I* as $\{1, 2, \ldots, m\}$. The y_i are restricted in the manner defined above. We now apply this definition to *PA*.

The dual of *PA* is

$$\text{Infimum}\left(\sum_{k\in k}\oplus' \sum_{i\in I_k} b_i y_{ki}\right)$$

$$\text{DA:}\quad \text{subject to:}\quad \sum_{k\in K}\oplus \sum_{i\in I_k} a_{kij} y_{ki} \geqslant c_j \text{ for all } j\in J$$

with the y_i suitably restricted as defined above.

Alternatively the primal model P could be expressed as one of finding an infimum. Then the dual D could be expressed as one of finding a supremum with the sign conventions for the dual variables on '$\leqslant$', '$<$', '$\geqslant$' and '$>$' reversed and the roles of '$\oplus$' and '$\oplus'$' interchanged.

5. The duality theorem

It is convenient first to prove the duality theorem for an LLP in the disjunctive form *PA* as a Lemma. We later extend it to an LLP in any form.

LEMMA. *One of the following three possibilities holds:*

(i) PA and DA are both feasible and have the same optimal objective values.

(ii) One of the PA and DA is infeasible and the other is infeasible or unbounded.

(iii) The constraints $\bigwedge_{i\in I_k} \sum_j a_{kij} x_j \cdot q_{ki} \cdot b_{ki}, \quad x_j \in \mathbb{R}^+$

from PA are infeasible for a proper subset of K as well as the corresponding (dual) constraints

$$\sum_{i\in I_k} a_{kij} y_{ki} \geqslant c_j \text{ for all } j\in J$$

from DA with the y_{ki} restricted as in DA but the LLP is solvable when restricted to the clauses in the complement of this subset of K.

(i) and (ii) correspond to the situation in LP.

Proof. We consider each value of $k\in K$ in turn and solve PA with the corresponding conjunction ('clause') of constraints.

If for a particular *k*, the inequalities are all non-strict, we have an LP model and one of the following possibilities occurs:

(a) The model is solvable with optimal objective value z_k, for the primal and corresponding dual objectives and optimal solution values $\underline{x}^{(k)}$ and $\underline{y}^{(k)}$ for the primal and dual.

(b) The model is unbounded and the corresponding dual model is feasible.

(c) The model is infeasible and the corresponding dual model is also infeasible.

If, for a particular *k*, some of the inequalities are strict we replace '$<$' inequalities by '⩽' subtracting ϵ from the RHS and '>' inequalities by '⩾' adding ϵ to the RHS where ϵ is positive. The resultant LP model and its dual are then solved (as a function of ϵ). As $\epsilon \to 0$ (but remains positive) the result will be (a), (b) or (c). In case (a) some of the optimal values of $\underline{x}^{(k)}$ and $\underline{y}^{(k)}$ will involve positive linear combinations of ϵ and some negative linear combinations. In the former case we replace them by their suprema and in the latter case by their infima.

For the full model PA if case (b) occurs for any k, PA is unbounded and DA infeasible (case (ii)). If case (c) occurs for some, but not all k and case (a) for all other k then PA is solvable but DA is infeasible (case (iii)). If case (c) occurs for all k then both PA and DA are infeasible (case (ii)). Otherwise the optimal objective value of PA is

$$\underset{k \in K}{\text{Maximum}}\ z_k = \sum_{k \in K} {}_{\oplus'} \left(\text{Infimum} \sum_{i \in I_k} b_{ki} y_{ki} \right) = \text{Infimum} \sum_{k \in K} {}_{\oplus'} \sum_{i \in I_k} b_{ki} y_{ki}$$

$$\text{subject to} \quad \sum_{i \in I_k} a_{kij} y_{ki} \geqslant c_j \ \text{ for all } k \in K,\ j \in J$$

$$\text{i.e.} \quad \text{subject to} \quad \underset{k \in K}{\text{Min}} \left(\sum_{i \in I_k} a_{kih} y_{ki} \geqslant c_j \right) \ \text{for all } j \in J$$

$$\text{i.e.} \quad \text{subject to} \quad \sum_{k \in K} {}_{\oplus} \sum_{i \in I_k} a_{kij} y_{ki} \geqslant c_j \ \text{ for all } j \in J$$

i.e. case (i) holds. □

We now state and prove the duality theorem for a general LLP.

THEOREM. *If P and D are both solvable (i.e. not infeasible or unbounded) they have the same optimal objective values.*

Proof. If P involves one (strict or non-strict) constraint then the result holds trivially. Otherwise the form L of the constraints of P are a compound proposition of one of the forms:

(a) $L_1 \wedge L_2$
(b) $L_1 \vee L_2$
(c) $\sim L_1$

where L_1, L_2 are Boolean functions.

We assume inductively that the theorem holds with L replaced by L_1 or L_2 alone. We refer to the corresponding models as $P(L_1)$, $P(L_2)$, $D(L_1)$ and $D(L_2)$ respectively.

$$\text{Case (a):} \quad L_\ell \left(\sum_{j \in J} a_{ij} x_j \cdot \rho_i \cdot b_i \right), \quad i \in I_\ell$$

can each be replaced by a set of linear (in)equalities defining the convex hull of feasible solutions. The result then follows from the duality theorem of LP.

Case (b): If $P(L)$ is solvable then $P(L_1)$, $P(L_2)$, $D(L_1)$ and $D(L_2)$

must all be solvable by the results of the lemma putting L_1 and L_2 in DNF.

$$z_{L_1 \vee L_2} = \mathrm{Max}(z_{L_1}, z_{L_2}) = \mathrm{Max}(z_{F_1}, z_{F_2}) = x_{F_1} \oplus' z_{F_2}$$
$$\text{subject to:} \quad F_1(a_{1j}y_1, y_2, \ldots, a_{m_i j}y_m) \geqslant c_j$$
$$\text{and} \quad F_2(a_{m_1+1j}y_1, a_{m_1+2j}y_2, \ldots, a_{m_1+m_2 j}y_{m_1+m_2}) \geqslant c_j \text{ for all } j \in J$$

i.e. subject to:

$$F_1(a_{1j}y_1, a_{2j}y_2, \ldots, a_{m_1 j}y_m) \oplus F_2(a_{m_1+1j}y_1, a_{m_2+2j}y_{m_1}, \ldots, a_{m_1+m_2 j}y_{m_1+m_2}) \geqslant c_j, \text{ for all } j \in J$$

where $I_1 = \{1, 2, \ldots, m_1\}$, $I_2 = \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$.

Hence $z_{L_1 \vee L_2} = z_{F_1 \oplus F_2}$.

In case (c) we can use De Morgan's laws to 'push' the negations down to the level of individual constraints. The result then holds by virtue of cases (a) and (b).

The logical formulation D can be generalised to allow atomic constraints involving generalised addition ($\oplus$ and $\oplus'$) as well as conventional addition (and subtraction). If we consider the objective as one of finding a supremum we must confine the $\oplus$ operation (min) to the objective function and '$\geqslant$' and '$>$' constraints and the $\oplus'$ operation (max) to '$\leqslant$' and '$<$' constraints.

For simplicity we will take a model in DNF and will convert all constraints to the '$\leqslant$' form (possibly replacing variables by their suprema or infima). The model

$$\text{Supremum} \sum_{k}{}_{\oplus} \sum_{j} c_{jk}x_{jk}$$
$$\text{subject to:} \quad \bigvee_{\ell} \bigwedge_{i} \sum_{k}{}_{\oplus'} \sum_{j} a_{i\ell jk}x_{jk} \leqslant b_{i\ell} \text{ for all } i, \ell$$
$$x_{jk} \geqslant 0$$

then has dual

$$\text{infimum} \quad \sum_{\ell}{}_{\oplus'} \sum_{i} b_{i\ell}y_{i\ell}$$
$$\text{subject to:} \quad \sum_{\ell}{}_{\oplus} \sum_{i} a_{i\ell jk}y_{i\ell} \geqslant c_{jk} \text{ for all } j, k$$
$$y_{i\ell} \geqslant 0 .$$

The duality theorem given above then still holds.

6. Duals of illustrative examples

EXAMPLE 1.

$$\text{Infimum} \quad ((8y_1 - 2y_2') \oplus' (8y_3 - 6y_4'))$$
$$\text{subject to:} \quad (2y_1 - y_2') \oplus (3y_3 - 4y_4') \geqslant 1$$

F1.1

$$(3y_1 + 2y_2') \oplus (4y_3 + y_4') \geqslant 2$$
$$y_1, y_2', y_3, y_4' \geqslant 0$$

y_2'' and y_4'' represent the infima of y_2 and y_4 respectively with the restriction that if these infima are strictly positive then y_2 and y_4 do not attain these values.

The optimal solution to E1.1 is

$$x_1' \text{ (i.e. infimum } (x_1)) = \frac{22}{7}$$

$$x_2'' \text{ (i.e. supremum } (x_2)) = \frac{4}{7}$$

$$\text{Objective} = \frac{30}{7}$$

The optimal solution to F1.1 is

$$y_1 = \frac{4}{7}$$

$$y_2' \text{ (i.e. infimum } (y_2)) = \frac{1}{7}$$

y_3 and y_4' (i.e. infimum (y_4)) can take any value in the polytope with vertices $(\frac{1}{2}, 0)$, $(\frac{9}{19}, \frac{2}{19})$, $(\frac{15}{28}, 0)$ and $(\frac{39}{49}, \frac{14}{49})$.

$$\text{Objective} = \frac{30}{7}$$

The dual of E1.2 is similar to F1.1 but y_2 and y_4 are negated.

The dual of E1.3 is

$$\text{Infimum} \quad ((8y_1 \oplus 8y_2) + (8y_3 \oplus' - 6y_4') + (-2y_5' \oplus' 8y_6) + (-2y_7' \oplus' 6y_8'))$$
$$\text{subject to:} \quad (2y_1 \oplus 3y_2) + (2y_3 \oplus -4y_4') + (-y_5' \oplus 3y_6) + (-y_7' \oplus -4y_8') \geqslant 1$$
$$\text{F1.3} \quad (3y_1 \oplus 4y_2) + (3y_3 \oplus y_4') + (2y_5' \oplus 4y_6) + (2y_7' \oplus y_8') \geqslant 2$$
$$y_1, y_2, y_3, y_4', y_5', y_6, y_7', y_8' \geqslant 0$$

with the restriction that if any of y_4, y_5, y_7, y_8 are strictly positive they do not attain their infima.

The optimal solution to E1.3 is the same as for E1.1.

F1.3 has alternative optimal solutions (essentially equivalent to that for F1.1 with variables renamed and duplicated).

$y_2' + y_6$ and $y_4' + y_8'$ can take any values in the polytope defined for y_3 and y_4' in F1.1

$$\text{Objective} = \frac{30}{7}$$

EXAMPLE 2.
The dual is

$$\text{Minimise} \quad ((10y_1 \oplus' 11y_2) + (6y_3 \oplus' 20y_4))$$
$$\text{subject to:} \quad (2y_1 \oplus 3y_2) + (y_3 \oplus 4y_4) \geqslant 1$$
$$\text{F2.1} \qquad (3y_1 \oplus -y_2) + (2y_3 \oplus y_4) \geqslant 1$$
$$y_1, y_2, y_3, y_4 \geqslant 0$$

The optimal solution of E2.1 (and E2.2) is

$$x_1 = 0,\ x_2 = 20,\ \text{objective} = 20$$

The optimal solution of F2.1 is

$$y_1 = 0,\ y_2 = 0,\ y_3 = 0,\ y_4 = 1,\ \text{objective} = 20$$

The dual of E2.2 is

$$\text{Minimise} \quad ((10y_1 + 6y_2) \oplus' (10y_3 + 20y_4) \oplus (11y_5 + 6y_6) \oplus (11y_7 + 20y_8))$$
$$\text{Subject to:} \quad (2y_1 + y_2) \oplus (2y_3 + 4y_4) \oplus (3y_5 + y_6) \oplus (3y_7 + 4y_8) \geqslant 1$$
$$\text{F2.2} \qquad (3y_1 + 2y_2) \oplus (3y_3 + y_4) \oplus (-y_5 + 2y_6) \oplus (-y_7 - y_8) \geqslant 1$$
$$y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \geqslant 0$$

This has alternatives, but essentially the same, optimal solution as F2.1 with variables renamed and duplicated such that

$$y_1 = \tfrac{1}{2},\ y_2 = 0,\ y_3 = \tfrac{1}{2},\ y_4 = 0,\ y_5 = \tfrac{1}{7},\ y_6 = \tfrac{4}{7},\ y_7 = 0\ ,$$
$$\text{Objective} = 20$$

EXAMPLE 3.
The dual E3.1 is

$$\text{Minimise} \quad ((-y_1 - y_2) \oplus (2y_3 + y_4))$$
$$\text{Subject to:} \quad (-2y_1 + y_2) \oplus (2y_3 + y_4) \geqslant 1$$
$$\text{F3.1} \qquad -y_1 \oplus (y_3 - y_4) \geqslant 1$$
$$y_1, y_2, y_3, y_4 \geqslant 0$$

The optimal solution of E3.1 (and E3.2) is

$$x_1 = 0, x_2 = 2, \text{Objective} = 2$$

F3.1 is infeasible

The dual of E3.2 is

$$\begin{array}{lll} & \text{Minimise} & ((-y_1 \oplus' 2y_2) + (-y_3 \oplus' 2y_4) + (-y_5 \oplus' 2y_6) + (-y_7 \oplus' y_8)) \\ & \text{Subject to:} & (-2y_1 \oplus 2y_2) + (-2y_3 \oplus y_4) + (y_5 \oplus 2y_6) + (y_7 \oplus y_8) \geqslant 1 \\ \text{F3.2} & & (y_1 \oplus y_2) + (y_3 \oplus -y_4) + y_6 - y_8 \geqslant 1 \\ & & y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \geqslant 0 \end{array}$$

which is again infeasible.

7. Generalised connectives

All Boolean functions can be expressed using the complete set of connectives $\{\wedge, \vee, \sim\}$. Other connectives can also be mapped into arithmetic functions to enable duals to LLPs to be constructed.

$A_1 \rightarrow A_2$ corresponds to $\text{Min}\,(-a_1, a_2)$

$A_1 \underset{(\leftrightarrow)}{\equiv} A_2$ corresponds to $-|a_1 + a_2|$

$A_1 \downarrow A_2$ (nor) corresponds to $-a_1 - a_2$

$A_1 | A_2$ (nand) corresponds to $\text{Min}\,(-a_1, -a_2)$

It is shown in McKinnon and Williams [7] that any LLP can conveniently be represented using a nested expression of ‘ge’ predicates applied to LP constraints. The predicate

ge$(r : P_1, P_2, \ldots, P_n)$ means ‘at most r of $P_1, P_2, \ldots, P_n$ are true’

If P_i corresponds to the constraint $\Sigma_j\, a_{ij}x_j \cdot\leqslant\cdot b_i$ the dual arithmetic function corresponding to ge$(r : P_1, P_2, \ldots, P_n)$ is

$$\sum_{\substack{(i_1, i_2, \ldots, i_r)\\ \in\theta_r}} {}_{\oplus}\, (a_{i1} + a_{i2} + \cdots + a_{ir})$$

where θ_r is the set of all permutations of r elements from $\{1, 2, \ldots, n\}$.

Defining this function as F_r we can then apply it in a nested fashion to define the dual of LLP expressed using nested ge predicates.

For example if we have the model

$$\begin{array}{ll} \text{Supremum} & \sum_{j\in J} c_j x_j \\ \text{subject to:} & (P_1 \wedge P_2) \leftrightarrow (P_3 \wedge P_4) \\ & x_j \geqslant 0 \end{array}$$

where P_i are linear (in)equalities the constraint can be written as

$$\begin{array}{l} \text{ge}(2 : \text{ge}(1 : \sim P_1, \sim P_2, \text{ge}(2 : P_3, P_4)) \\ \qquad \text{ge}(1 : \sim P_3, \sim P_4, \text{ge}(2 : P_1, P_2))) \end{array}$$

The dual model would then be

$$\begin{aligned}
&\text{Infimum} \quad F_2'(F_1'(-b_1y_1'', -b_2y_2'', F_2'(b_3y_3'', b_4y_4''')),\\
&\text{subject to:} \quad F_1'(-b_3y_3'', -b_4y_4'', F_2'(b_1y_1'', b_2y_2'')))\\
&\qquad F_2(F_1(-a_{1j}y_1'', -a_{2j}y_2'', F_2(a_{3j}y_3'', a_{4j}y_4'')),\\
&\qquad\quad F_1(-a_{3j}y_3'', -a_{4j}y_4'', F_2(a_{1j}y_1'', a_{2j}y_2''))) \geqslant c_j \text{ all } j \in J\\
&\qquad y_1'', y_2'', y_3'', y_4'' \geqslant 0
\end{aligned}$$

F_1' and F_2' are F_1 and F_2 with the operation $\oplus$ replaced by $\oplus'$.

8. Relations with other work

The idea of disjunctive constraints goes back to Balas [1] who in Balas [2] defines the dual of a Disjunctive Programme as an LP. This is done by expressing it in DNF and converting the 'Minimax' objective of DA into a linear form. The constraints of DA are dealt with by repeating all the linear expressions which are arguments of $\Sigma_\oplus$. A regularity condition is imposed to disallow models in DNF where one, but not all conjunctive clauses are both primal and dual infeasible with at least one primal clause leading to a finite optimal solution. In this way the full correspondence with LP duality is preserved.

It is pointed out in Williams [8] that by taking the dual of this dual formulation one obtains an alternative to the traditional MIP formulation of a disjunction. This formulation again goes back to Balas [1] and is explained in Jeroslow [6]. If the full disjunctive formulation is given then a 'sharp' formulation is obtained guaranteeing an integer solution to an otherwise LP model.

The regularity condition mentioned above corresponds to the condition of demanding that all the polytopes corresponding to the conjunctive clauses, when the model is in DNF, have the same recession directions. This is shown by Jeroslow [6] to be a necessary and sufficient condition for a Disjunctive Programme to have a MIP formulation.

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